

On Holant Theorem and Its Proof

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Abstract

Holographic algorithms are a recent breakthrough in computer science and has found applications in information theory. This paper provides a proof to the central component of holographic algorithms, namely, the Holant theorem. Compared with previous works, the proof appears simpler and more direct. Along the proof, we also develop a mathematical tool, which we call c-tensor. We expect the notion of c-tensor may be applicable over a wide range of analysis.

I. INTRODUCTION

Holographic algorithms [1], [2], [3], [4], pioneered by L. Valiant, are a recent breakthrough in the theory of computation. Specifically, in his landmark paper [1], Valiant presented polynomial-time algorithms – holographic algorithms — for a number of problem families which were not known to be in P previously. Holographic algorithms have been recently introduced to the information theory community and applied to solving the capacity of certain constrained coding problems [5].

Briefly, holographic algorithms set out to compute the sum of a multi-variate function over its configuration space where the function considered involves a large number of variables and factors as the product of local functions. Problems of such nature arise frequently in information theory, for example in computing constrained-coding capacity and in decoding various error correction codes. It is well-known that in general, obtaining exact solutions for such problems is computationally intractable. However, in the methodology of holographic algorithms, when it is possible to apply certain transformation of the problems, polynomial-time solvers can be constructed. Such transformations, which Valiant refers to as holographic reductions, form the basis of holographic algorithms.

The central component of holographic reductions is the *Holant Theorem*, which was introduced and proved in [1]. The proof however appears rather encrypted to many audience, and subsequently inspired an alternative proof given by Cai and Choudhary in [6]. As the proof of [6] uses a sophisticated machinery of tensors that is not familiar to broad audience, here we present a more direct proof. At least to us, our proof seems simpler, accessible for general audience and contains more insights. Additionally, the Holant Theorem in this paper is proved in its most general form, namely, the variables may take values from arbitrary alphabet. This contrasts the previous works where only binary alphabets are considered.

It is remarkable that our proof relies on the notion of “c-tensor”, a term which we coin in this paper. In a sense equivalent to the standard tensor product, c-tensor differs from the standard tensor product in that it is a commutative operation. Using c-tensor, our proof of Holant Theorem appears more transparent. Although perhaps under the guise of mathematical literature, the notion of c-tensor synthesized in this paper appears to be a useful tool. We expect that c-tensor may find other applications in context beyond Holant Theorem.

This paper is organized as follows. Section II gives mathematical preliminaries, where the main focus is to develop the mathematical tool of c-tensor. Section III states and proves Holant theorem. Section IV provides a brief conclusion.

II. PRELIMINARIES

A. Assignments

For an arbitrary finite set¹ E and any finite alphabet \mathcal{A} , we refer to any function mapping E into \mathcal{A} as an \mathcal{A} -assignment on E , and collectively denote the set of all such assignments by \mathcal{A}^E . If x is an \mathcal{A} -assignment on E , we often write x_E in place of x to make explicit the domain of function x . For any subset $U \subseteq E$, the *restriction* of an \mathcal{A} -assignment x_E to U will be denoted by $x_{E:U}$. That is, $x_{E:U}$ is an \mathcal{A} -assignment on U such that for every $e \in U$, $x_{E:U}(e) = x_E(e)$. If E is clear from the context, we may

¹All sets in this work are non-empty unless otherwise specified.

write x_U in place of $x_{E:U}$ for simplicity. In particular, such practice will be more common when U is a singleton $\{e\}$ for some $e \in E$. In this case, we always write $x_{\{e\}}$ rather than $x_{E:\{e\}}$.

With a slight abuse of notation, an assignment, say x_E , is also treated as a *set*, namely, the set $\{(e, x_E(e)) | e \in E\}$ or the graph of function x_E . Under such interpretation, restriction $x_{E:U}$ is also understood as set $\{(e, x_E(e)) | e \in U\}$. In addition, for any two disjoint finite sets E_1 and E_2 and any two \mathcal{A} -assignments x_{E_1} and x_{E_2} , the union $x_{E_1} \cup x_{E_2}$ is well defined and can be interpreted back as an \mathcal{A} -assignment on $E_1 \cup E_2$ defined by

$$(x_{E_1} \cup x_{E_2})(e) = \begin{cases} x_{E_1}(e), & \text{if } e \in E_1 \\ x_{E_2}(e), & \text{if } e \in E_2 \end{cases}$$

Conversely, every \mathcal{A} -assignment $x_{E_1 \cup E_2}$ on $E_1 \cup E_2$ can be understood as the union $x_{E_1} \cup x_{E_2}$ of two \mathcal{A} -assignments x_{E_1} and x_{E_2} , which are the restrictions of $x_{E_1 \cup E_2}$ to E_1 and E_2 , respectively. Furthermore, it is easy to verify that the decomposition of any $x_{E_1 \cup E_2}$ in terms of the union of two such restrictions is unique for any fixed choice of E_1 and E_2 . This establishes a one-to-one correspondence between the set $\mathcal{A}^{E_1 \cup E_2}$ of all \mathcal{A} -assignments on $E_1 \cup E_2$ and the cartesian product $\mathcal{A}^{E_1} \times \mathcal{A}^{E_2}$. Extending this argument by induction (on the number of disjoint sets), we obtain the following lemma.

Lemma 1: Let E be a finite set and $\{E_1, \dots, E_p\}$ be an arbitrary partition of E . Then for any finite alphabet \mathcal{A} and any \mathcal{A} -assignment x_E on E , there exists a unique sequence of restrictions $x_{E:E_1}, x_{E:E_2}, \dots, x_{E:E_p}$ of x_E to E_1, E_2, \dots, E_p respectively such that $x_E = \bigcup_{i=1}^p x_{E:E_i}$. Conversely, for any given \mathcal{A} -assignments $x_{E_1}, x_{E_2}, \dots, x_{E_p}$, the union $\bigcup_{i=1}^p x_{E_i}$ is an \mathcal{A} -assignment on E . That is, there is a one-to-one correspondence between \mathcal{A}^E and the p -fold cartesian product $\mathcal{A}^{E_1} \times \mathcal{A}^{E_2} \times \dots \times \mathcal{A}^{E_p}$.

B. Space of functions

In this paper, we will often work with vector spaces in the form of \mathbb{C}^S , where \mathbb{C}^S is the set of all functions from a finite set S to the field of complex numbers \mathbb{C} . The following lemma, straight-forward to prove, justifies that the set \mathbb{C}^S is a vector space.

Lemma 2: For any finite set S , let the set \mathbb{C}^S of all functions mapping S into \mathbb{C} be equipped with the following two operations:

- Addition: $\forall f, f' \in \mathbb{C}^S, (f + f')(s) := f(s) + f'(s)$ for every $s \in S$.
- Scalar multiplication: $\forall f \in \mathbb{C}^S$ and $\alpha \in \mathbb{C}, (\alpha f)(s) := \alpha(f(s))$ for every $s \in S$.

Then \mathbb{C}^S is a vector space isomorphic to $\mathbb{C}^{|S|}$.

Note that from this lemma it is immediate that $\mathbb{C}^S \cong \mathbb{C}^{S'}$ whenever $|S| = |S'| < \infty$. In the lemma, the fact that $\mathbb{C}^S \cong \mathbb{C}^{|S|}$ can be established by advising an explicit vector space isomorphism. Such isomorphism is clearly not unique. Let r be an arbitrary set bijection from S into the set $\{1, \dots, |S|\}$. It is not hard to show that $\sigma : \mathbb{C}^S \rightarrow \mathbb{C}^{|S|}$ such that

$$\sigma(f) = (f(r^{-1}(1)), f(r^{-1}(2)), \dots, f(r^{-1}(|S|)))$$

for all $f \in \mathbb{C}^S$ is a vector space isomorphism. We refer to such isomorphism as a *natural* one. At places where confusion is unlikely, we may write $f r^{-1}(\cdot)$ for the composition $f(r^{-1}(\cdot))$ to enhance readability.

Since \mathbb{C}^S is a vector space over \mathbb{C} of dimension $|S|$, one can find a set of functions which form a basis in \mathbb{C}^S . An obvious example of a basis is the “standard” one, namely, the set $\mathcal{B}_{\text{std}} = \{\delta_s : s \in S\}$ where for any $s \in S$, $\delta_s : S \rightarrow \mathbb{C}$ is such that $\delta_s(x) = 1$ if $x = s$ and otherwise $\delta_s(x) = 0$ for all $x \in S$. Note that a natural vector space isomorphism mapping \mathbb{C}^S to $\mathbb{C}^{|S|}$ simply accounts to taking coordinates with respect to \mathcal{B}_{std} under the ordering induced by the bijection r associated with the natural isomorphism. More explicitly, let $\sigma : \mathbb{C}^S \rightarrow \mathbb{C}^{|S|}$ be a natural isomorphism and let $r : S \rightarrow \{1, \dots, |S|\}$ be the bijection associated with σ . Then for any $f \in \mathbb{C}^S$, the i th component of $\sigma(f)$ is the coefficient of $\delta_{r^{-1}(i)}$ when f is expressed as a linear combination of elements from the basis \mathcal{B}_{std} . Another example of basis can be constructed as follows. Let $S = \{s_1, \dots, s_{|S|}\}$ and let $S_i = \{s_1, \dots, s_i\}$ for all $1 \leq i \leq |S|$. Further let $\mathcal{B} = \{\beta_1, \dots, \beta_{|S|}\}$ where $\beta_i : S \rightarrow \mathbb{C}$ is such that $\beta_i(x) = 1$ if $x \in S_i$ and otherwise $\beta_i(x) = 0$ for all $x \in S$. Then, one can easily verify that the set \mathcal{B} is a basis of \mathbb{C}^S .

Finally, for any vector space \mathbb{C}^S , as above, we define the map $\langle \cdot, \cdot \rangle : \mathbb{C}^S \times \mathbb{C}^S \rightarrow \mathbb{C}$ by

$$\langle f, f' \rangle := \sum_{x \in S} f(x) f'(x) \quad (1)$$

for all $f, f' \in \mathbb{C}^S$. It is clear that the map $\langle \cdot, \cdot \rangle$ is bilinear.

As we will see momentarily, vector space \mathbb{C}^S considered in this paper mostly takes S as \mathcal{A}^E for some choice of alphabet \mathcal{A} and finite set E . In this case, we will write $\mathbb{C}_{\mathcal{A}}^E$ in place of $\mathbb{C}^{(\mathcal{A}^E)}$ to lighten the notations.

C. *c-Tensors*

Let E_1 and E_2 be two disjoint sets. For any $f_1 \in \mathbb{C}_{\mathcal{A}}^{E_1}$ and $f_2 \in \mathbb{C}_{\mathcal{A}}^{E_2}$, we define the *c-tensor* $f_1 \odot f_2$ of f_1 and f_2 as the element in $\mathbb{C}_{\mathcal{A}}^{E_1 \cup E_2}$ (that is, the function mapping $\mathcal{A}^{E_1 \cup E_2}$ into \mathbb{C}) such that for every $x_{E_1 \cup E_2} \in \mathcal{A}^{E_1 \cup E_2}$,

$$(f_1 \odot f_2)(x_{E_1 \cup E_2}) := f_1(x_{(E_1 \cup E_2):E_1}) f_2(x_{(E_1 \cup E_2):E_2})$$

Inductively, we can extend the notion of c-tensor to arbitrary number of functions $f_1 \in \mathbb{C}_{\mathcal{A}}^{E_1}, f_2 \in \mathbb{C}_{\mathcal{A}}^{E_2}, \dots, f_p \in \mathbb{C}_{\mathcal{A}}^{E_p}$, where $\{E_1, E_2, \dots, E_p\}$ is an arbitrary collection of disjoint finite sets. The following lemma is directly provable from the definition of c-tensor.

Lemma 3: c-tensor is commutative and associative.

Proof: Let $f_1 \in \mathbb{C}_{\mathcal{A}}^{E_1}, f_2 \in \mathbb{C}_{\mathcal{A}}^{E_2}$ and $f_3 \in \mathbb{C}_{\mathcal{A}}^{E_3}$ where E_1, E_2 and E_3 are finite disjoint sets. Then,

$$\begin{aligned} (f_1 \odot f_2) \odot f_3(x_{E_1 \cup E_2 \cup E_3}) &= f_1 \odot f_2(x_{E_1 \cup E_2 \cup E_3:E_1 \cup E_2}) f_3(x_{E_1 \cup E_2 \cup E_3:E_3}) \\ &= f_1(x_{E_1 \cup E_2 \cup E_3:E_1}) f_2(x_{E_1 \cup E_2 \cup E_3:E_2}) f_3(x_{E_1 \cup E_2 \cup E_3:E_3}) \\ &= f_1(x_{E_1 \cup E_2 \cup E_3:E_1}) f_2 \odot f_3(x_{E_1 \cup E_2 \cup E_3:E_2 \cup E_3}) \\ &= f_1 \odot (f_2 \odot f_3)(x_{E_1 \cup E_2 \cup E_3}) \end{aligned}$$

Further,

$$\begin{aligned}
f_1 \odot f_2(x_{E_1 \cup E_2}) &= f_1(x_{E_1 \cup E_2 : E_1}) f_2(x_{E_1 \cup E_2 : E_2}) \\
&= f_2(x_{E_1 \cup E_2 : E_2}) f_1(x_{E_1 \cup E_2 : E_1}) \\
&= f_2 \odot f_1(x_{E_1 \cup E_2})
\end{aligned}$$

As claimed. ■

As will be shown momentarily, c-tensor of two functions is in a sense equivalent to the standard notion of tensor product of two vectors. The reason we refer to this operation “c-tensor” is to emphasize its commutative nature, which in general does not hold for the standard tensor product. Since the c-tensor is independent of bracketing and ordering (due to the previous lemma), the notation $\bigodot_{i=1}^p f_i$, denoting the p -fold c-tensor of functions f_1, f_2, \dots, f_p , is well defined.

We now show the equivalence between c-tensor and tensor product, where standard tensor product of two vectors u and v is denoted by $u \otimes v$.

Lemma 4: Let E_1 and E_2 be disjoint and $\sigma_1 : \mathbb{C}_{\mathcal{A}}^{E_1} \rightarrow \mathbb{C}^{|\mathcal{A}^{E_1}|}$ and $\sigma_2 : \mathbb{C}_{\mathcal{A}}^{E_2} \rightarrow \mathbb{C}^{|\mathcal{A}^{E_2}|}$ be two natural vector-space isomorphisms. Then there exists a natural vector-space isomorphism $\sigma : \mathbb{C}_{\mathcal{A}}^{E_1 \cup E_2} \rightarrow \mathbb{C}^{|\mathcal{A}^{E_1 \cup E_2}|}$ such that $\sigma(f_1 \odot f_2) = \sigma_1(f_1) \otimes \sigma_2(f_2)$ for any $f_1 \in \mathbb{C}_{\mathcal{A}}^{E_1}$ and $f_2 \in \mathbb{C}_{\mathcal{A}}^{E_2}$.

Proof: Let $r_1 : \mathcal{A}^{E_1} \rightarrow \{1, \dots, |\mathcal{A}^{E_1}|\}$ and $r_2 : \mathcal{A}^{E_2} \rightarrow \{1, \dots, |\mathcal{A}^{E_2}|\}$ be the two bijections associated with σ_1 and σ_2 , respectively. Define the set map $r : \mathcal{A}^{E_1 \cup E_2} \rightarrow \{1, \dots, |\mathcal{A}^{E_1}|\} \times \{1, \dots, |\mathcal{A}^{E_2}|\}$ given by $r(x_{E_1 \cup E_2}) = (r_1(x_{(E_1 \cup E_2) : E_1}), r_2(x_{(E_1 \cup E_2) : E_2}))$. Then r is a bijection since r_1 and r_2 are bijections. Now define $\sigma : \mathbb{C}_{\mathcal{A}}^{E_1 \cup E_2} \rightarrow \mathbb{C}^{|\mathcal{A}^{E_1 \cup E_2}|}$ as

$$\sigma(f) = (f r^{-1}(1, 1), f r^{-1}(1, 2), \dots, f r^{-1}(|\mathcal{A}^{E_1}|, |\mathcal{A}^{E_2}|))$$

for all $f \in \mathbb{C}_{\mathcal{A}}^{E_1 \cup E_2}$. Then σ is a natural isomorphism and

$$\sigma(f_1 \odot f_2) = ((f_1 \odot f_2)(r^{-1}(1, 1)), (f_1 \odot f_2)(r^{-1}(1, 2)), \dots, (f_1 \odot f_2)(r^{-1}(|\mathcal{A}^{E_1}|, |\mathcal{A}^{E_2}|)))$$

$$\begin{aligned}
&= (f_1 r_1^{-1}(1) f_2 r_2^{-1}(1), f_1 r_1^{-1}(1) f_2 r_2^{-1}(2), \dots, f_1 r_1^{-1}(1) f_2 r_2^{-1}(|\mathcal{A}^{E_2}|), \\
&\quad f_1 r_1^{-1}(2) f_2 r_2^{-1}(1), f_1 r_1^{-1}(2) f_2 r_2^{-1}(2), \dots, f_1 r_1^{-1}(2) f_2 r_2^{-1}(|\mathcal{A}^{E_2}|), \\
&\quad \dots \\
&\quad f_1 r_1^{-1}(|\mathcal{A}^{E_1}|) f_2 r_2^{-1}(1), f_1 r_1^{-1}(|\mathcal{A}^{E_1}|) f_2 r_2^{-1}(2), \dots, f_1 r_1^{-1}(|\mathcal{A}^{E_1}|) f_2 r_2^{-1}(|\mathcal{A}^{E_2}|)) \\
&= (f_1 r_1^{-1}(1) \sigma_2(f_2), \dots, f_1 r_1^{-1}(|\mathcal{A}^{E_1}|) \sigma_2(f_2)) \\
&= \sigma_1(f_1) \otimes \sigma_2(f_2)
\end{aligned}$$

as desired. ■

Extending this lemma by induction to multi-fold c-tensor gives the following corollary.

Corollary 1: Let E_1, E_2, \dots, E_p be pairwise disjoint and $\sigma_1 : \mathbb{C}_{\mathcal{A}}^{E_1} \rightarrow \mathbb{C}^{|\mathcal{A}^{E_1}|}$, $\sigma_2 : \mathbb{C}_{\mathcal{A}}^{E_2} \rightarrow \mathbb{C}^{|\mathcal{A}^{E_2}|}$, \dots , $\sigma_p : \mathbb{C}_{\mathcal{A}}^{E_p} \rightarrow \mathbb{C}^{|\mathcal{A}^{E_p}|}$ be natural vector-space isomorphisms. Then there exists a natural vector-space isomorphism $\sigma : \mathbb{C}_{\mathcal{A}}^{\bigcup_i^p E_i} \rightarrow \mathbb{C}^{|\mathcal{A}^{\bigcup_i^p E_i}|}$ such that

$$\sigma\left(\bigodot_i^p f_i\right) = \sigma_1(f_1) \otimes \sigma_2(f_2) \otimes \dots \otimes \sigma_p(f_p)$$

for any $f_1 \in \mathbb{C}_{\mathcal{A}}^{E_1}, f_2 \in \mathbb{C}_{\mathcal{A}}^{E_2}, \dots, f_p \in \mathbb{C}_{\mathcal{A}}^{E_p}$.

This establishes a sense of equivalence between c-tensor and the standard tensor product.

D. Basis

Let \mathcal{A} be a finite alphabet and E be a finite set. For each $e \in E$, define map $\tau_e : \mathbb{C}^{\mathcal{A}} \rightarrow \mathbb{C}_{\mathcal{A}}^{\{e\}}$ such that for all $f \in \mathbb{C}^{\mathcal{A}}$

$$\tau_e(f)(x_{\{e\}}) = f(x_{\{e\}}(e))$$

for all $x_{\{e\}} \in \mathcal{A}^{\{e\}}$.

The following lemma shows that τ_e is an isomorphism.

Lemma 5: Let E be a finite set and for each $e \in E$, define $\tau_e : \mathbb{C}^{\mathcal{A}} \rightarrow \mathbb{C}_{\mathcal{A}}^{\{e\}}$ as above. Then each τ_e is a vector space isomorphism.

Proof: Let f and f' be arbitrary functions from $\mathbb{C}^{\mathcal{A}}$ and α and α' be arbitrary scalars from \mathbb{C} . Then for all $x_{\{e\}} \in \mathcal{A}^{\{e\}}$ we have

$$\begin{aligned}\tau_e(\alpha f + \alpha' f')(x_{\{e\}}) &= (\alpha f + \alpha' f')(x_{\{e\}}(e)) \\ &= (\alpha f)(x_{\{e\}}(e)) + (\alpha' f')(x_{\{e\}}(e)) \\ &= \alpha f(x_{\{e\}}(e)) + \alpha' f'(x_{\{e\}}(e)) \\ &= \alpha \tau_e(f)(x_{\{e\}}) + \alpha' \tau_e(f')(x_{\{e\}})\end{aligned}$$

Hence, $\tau_e(\alpha f + \alpha' f') = \alpha \tau_e(f) + \alpha' \tau_e(f')$ and therefore τ_e preserves addition and scalar multiplication.

Assume $f, f' \in \mathbb{C}^{\mathcal{A}}$ are such that $\tau_e(f) = \tau_e(f')$ then $\tau_e(f)(x_{\{e\}}) = \tau_e(f')(x_{\{e\}})$ for all $x_{\{e\}} \in \mathcal{A}^{\{e\}}$. Hence, $f(x_{\{e\}}(e)) = f'(x_{\{e\}}(e))$ for all $x_{\{e\}}$. This is equivalent to $f(a) = f'(a)$ for all $a \in \mathcal{A}$ and hence $f = f'$, which implies τ_e is injective. Surjectivity of τ_e is automatic since $\mathbb{C}^{\mathcal{A}}$ and $\mathbb{C}_{\mathcal{A}}^{\{e\}}$ both have the same finite dimension. ■

By lemma 2 we know that $\mathbb{C}^{\mathcal{A}}$ is a vector space of dimension $|\mathcal{A}|$. Let \mathcal{B} be a basis² of $\mathbb{C}^{\mathcal{A}}$ and for each $e \in E$ denote by $\tau_e(\mathcal{B})$ the image of \mathcal{B} under τ_e , that is, $\tau_e(\mathcal{B}) = \{\tau_e(\beta) | \beta \in \mathcal{B}\}$. Then $\tau_e(\mathcal{B})$ forms a basis in $\mathbb{C}_{\mathcal{A}}^{\{e\}}$ for all $e \in E$ because an isomorphism maps basis into basis. Since each element in the basis $\tau_e(\mathcal{B})$ is a map from $\mathcal{A}^{\{e\}}$ into \mathbb{C} , then for any \mathcal{B} -assignment on E , say b_E , the c-tensor $\bigodot_{e \in E} \tau_e(b_E(e))$ is a map from \mathcal{A}^E into \mathbb{C} . This motivates the following definition. For any E , \mathcal{A} and \mathcal{B} as above, we define the map $\mathcal{E} : \mathcal{A}^E \times \mathcal{B}^E \rightarrow \mathbb{C}$ such that

$$\mathcal{E}(a_E, b_E) = \left(\bigodot_{e \in E} \tau_e(b_E(e)) \right)(a_E)$$

for all $(a_E, b_E) \in \mathcal{A}^E \times \mathcal{B}^E$. Note that the name of the map was chosen to emphasize that its arguments are assignments on E . Hence in subsequent discussions- if we write for instance \mathcal{E}_i , then we mean the map from $\mathcal{A}^{E_i} \times \mathcal{B}^{E_i}$ to \mathbb{C} defined as above.

²Our definition for a basis coincide with the one from linear algebra, i.e, a basis is a linearly independent spanning set. In contrast, the definition of a basis in Valiant's work was that of a spanning set. This work still holds if the definition of a basis was chosen to confine with Valiant's definition.

Now if b_E is a fixed \mathcal{B} -assignment on E , then map \mathcal{E} induces, via b_E , a function $\mathcal{E}^{[b_E]}$ mapping \mathcal{A}^E to \mathbb{C} : Formally, for any $b_E \in \mathcal{B}^E$ we define $\mathcal{E}^{[b_E]} : \mathcal{A}^E \rightarrow \mathbb{C}$ by $\mathcal{E}^{[b_E]}(a_E) = \mathcal{E}(a_E, b_E)$ for all $a_E \in \mathcal{A}^E$.

Conversely, if a_E is a fixed \mathcal{A} -assignment on E , then map \mathcal{E} induces, via a_E , a function $\mathcal{E}_{[a_E]}$ mapping \mathcal{B}^E to \mathbb{C} : Formally, for any $a_E \in \mathcal{A}^E$ we define $\mathcal{E}_{[a_E]} : \mathcal{B}^E \rightarrow \mathbb{C}$ by $\mathcal{E}_{[a_E]}(b_E) = \mathcal{E}(a_E, b_E)$ for all $b_E \in \mathcal{B}^E$.

Before we proceed, we need the following result from algebra

Theorem 1: If $\mathcal{B} = \{\beta_1, \dots, \beta_k\}$ is a basis of \mathbb{C}^k , then $\{\beta_{i_1} \otimes \dots \otimes \beta_{i_m} \mid \beta_{i_1}, \dots, \beta_{i_m} \in \mathcal{B}, 1 \leq i_j \leq k \ \forall j\}$ is a basis of the vector space \mathbb{C}^{k^m} .

The following theorem is a recast of the previous one in the costumes of c-tensors and our frame of work.

Theorem 2: If $\mathcal{B} = \{\beta_1, \dots, \beta_{|\mathcal{A}|}\}$ is a basis of $\mathbb{C}^{\mathcal{A}}$, then $\{\mathcal{E}^{[b_E]} \mid b_E \in \mathcal{B}^E\}$ is a basis of $\mathbb{C}_{\mathcal{A}}^E$.

Proof: First note that $\{\{e_1\}, \dots, \{e_{|E|}\}\}$ forms a partition of E of size $|E|$. For each $e \in E$, let $\tau_e : \mathbb{C}^{\mathcal{A}} \rightarrow \mathbb{C}_{\mathcal{A}}^{\{e\}}$ be an isomorphism as in lemma 5. Further, let $\sigma : \mathbb{C}^{\mathcal{A}} \rightarrow \mathbb{C}^{|\mathcal{A}|}$ be a natural vector space isomorphism and define $\sigma_e : \mathbb{C}_{\mathcal{A}}^{\{e\}} \rightarrow \mathbb{C}^{|\mathcal{A}|}$ as the composition $\sigma_e = \sigma \circ \tau_e^{-1}$ for all $e \in E$. Then, σ_e is a natural vector space isomorphism for all $e \in E$. Hence, by corollary 1, there exists a (natural) isomorphism $\sigma_E : \mathbb{C}_{\mathcal{A}}^E \rightarrow \mathbb{C}^{|\mathcal{A}^E|}$ such that for all $f_1 \in \mathbb{C}_{\mathcal{A}}^{\{e_1\}}, \dots, f_{|E|} \in \mathbb{C}_{\mathcal{A}}^{\{e_{|E|}\}}$,

$$\sigma_E\left(\bigodot_{i=1}^{|E|} f_i\right) = \sigma_{e_1}(f_1) \otimes \dots \otimes \sigma_{e_{|E|}}(f_{|E|})$$

Let $\sigma(\mathcal{B}) = \{\tilde{\beta}_1, \dots, \tilde{\beta}_{|\mathcal{A}|}\}$ be the image of \mathcal{B} under σ , then $\sigma(\mathcal{B})$ is a basis of $\mathbb{C}^{|\mathcal{A}|}$. Let $X = \{\mathcal{E}^{[b_E]} \mid b_E \in \mathcal{B}^E\}$ and $Y = \{\tilde{\beta}_{i_1} \otimes \dots \otimes \tilde{\beta}_{i_{|E|}} \mid \tilde{\beta}_{i_1}, \dots, \tilde{\beta}_{i_{|E|}} \in \sigma(\mathcal{B}), 1 \leq i_j \leq |\mathcal{A}| \ \forall j\}$. Then theorem 1 asserts that Y is a basis of $\mathbb{C}^{|\mathcal{A}^E|}$. Hence we are done if we can show that $\sigma_E(X) = Y$, since this will imply that X is a basis (due to the fact that an isomorphism maps basis to basis). To this end, note that

$$\begin{aligned} x \in X &\Rightarrow x = \bigodot_{e \in E} \tau_e(b_E(e)), \text{ some } b_E \in \mathcal{B}^E \\ &\Rightarrow x = \tau_{e_1}(\beta_{i_1}) \odot \dots \odot \tau_{e_{|E|}}(\beta_{i_{|E|}}), \text{ some } \beta_{i_1}, \dots, \beta_{i_{|E|}} \in \mathcal{B} \\ &\Rightarrow \sigma_E(x) = \sigma_{e_1}(\tau_{e_1}(\beta_{i_1})) \otimes \dots \otimes \sigma_{e_{|E|}}(\tau_{e_{|E|}}(\beta_{i_{|E|}})), \text{ some } \beta_{i_1}, \dots, \beta_{i_{|E|}} \in \mathcal{B} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \sigma_E(x) = \sigma(\beta_{i_1}) \otimes \dots \otimes \sigma(\beta_{i_{|E|}}), \text{ some } \beta_{i_1}, \dots, \beta_{i_{|E|}} \in \mathcal{B} \\
&\Rightarrow \sigma_E(x) = \tilde{\beta}_{j_1} \otimes \dots \otimes \tilde{\beta}_{j_{|E|}}, \text{ some } \tilde{\beta}_{j_1}, \dots, \tilde{\beta}_{j_{|E|}} \in \sigma(\mathcal{B}) \\
&\Rightarrow \sigma_E(x) \in Y
\end{aligned}$$

Thus, $\sigma_E(X) \subseteq Y$. Since both σ_E and σ are bijective, it follows that $|\sigma_E(X)| = |X| = |\mathcal{B}|^{|E|} = |\sigma(\mathcal{B})|^{|E|} = |Y|$. Therefore, $\sigma_E(X) = Y$. ■

The following definition is fundamental

Definition 1: Let \mathcal{B} be a basis for $\mathbb{C}^{\mathcal{A}}$ and for any $f \in \mathbb{C}_{\mathcal{A}}^E$ define

- \hat{f} as the unique³ map from \mathcal{B}^E into \mathbb{C} satisfying for all $a \in \mathcal{A}^E$

$$f(a) = \langle \hat{f}, \mathcal{E}_{[a]} \rangle$$

- \check{f} as the map from \mathcal{B}^E into \mathbb{C} such that for all $b \in \mathcal{B}^E$,

$$\check{f}(b) = \langle f, \mathcal{E}^{[b]} \rangle$$

The following proposition shows that the definition of \hat{f} makes sense.

Proposition 1: For any $f \in \mathbb{C}_{\mathcal{A}}^E$ and any basis \mathcal{B} of $\mathbb{C}^{\mathcal{A}}$, \hat{f} exists and is unique.

Proof: Theorem 2 asserts that $\{\mathcal{E}^{[b]} | b \in \mathcal{B}^E\}$ is a basis of $\mathbb{C}_{\mathcal{A}}^E$. Hence, f can be uniquely written as $f = \sum_{b \in \mathcal{B}^E} c_b \mathcal{E}^{[b]}$ where $c_b \in \mathbb{C}$ for all $b \in \mathcal{B}^E$. Let $\hat{f}(b) = c_b$ then \hat{f} is a map from \mathcal{B}^E into \mathbb{C} and it is unique. Therefore, we are done if we show this \hat{f} is consistent with the definition. But $f = \sum_{b \in \mathcal{B}^E} \hat{f}(b) \mathcal{E}^{[b]}$ implies

$$f(a) = \sum_{b \in \mathcal{B}^E} \hat{f}(b) \mathcal{E}^{[b]}(a) = \sum_{b \in \mathcal{B}^E} \hat{f}(b) \mathcal{E}(a, b) = \sum_{b \in \mathcal{B}^E} \hat{f}(b) \mathcal{E}_{[a]}(b) = \langle \hat{f}, \mathcal{E}_{[a]} \rangle$$

as desired. ■

Let E_1 and E_2 be two disjoint finite sets and assume $f_1 \in \mathbb{C}_{\mathcal{A}}^{E_1}$ and $f_2 \in \mathbb{C}_{\mathcal{A}}^{E_2}$. Let $E = E_1 \cup E_2$, then

³If \mathcal{B} was a spanning set, then \hat{f} is defined as any function in $\mathbb{C}_{\mathcal{A}}^E$ satisfying $f(a) = \langle \hat{f}, \mathcal{E}_{[a]} \rangle$ for all $a \in \mathcal{A}^E$.

for any $b_E \in \mathcal{B}^E$, we have

$$\begin{aligned}
\langle f_1 \odot f_2, \mathcal{E}^{[b_E]} \rangle &= \sum_{a_E \in \mathcal{A}^E} (f_1 \odot f_2)(a_E) \mathcal{E}(a_E, b_E) \\
&= \sum_{a_E \in \mathcal{A}^E} (f_1 \odot f_2)(a_E) \left(\bigodot_{e \in E} \tau_e(b_E(e)) \right)(a_E) \\
&= \sum_{a_E \in \mathcal{A}^E} f_1(a_{E:E_1}) f_2(a_{E:E_2}) \left(\bigodot_{e \in E_1} \tau_e(b_E(e)) \right) \odot \left(\bigodot_{e \in E_2} \tau_e(b_E(e)) \right)(a_E) \\
&= \sum_{a_E \in \mathcal{A}^E} f_1(a_{E:E_1}) f_2(a_{E:E_2}) \left(\bigodot_{e \in E_1} \tau_e(b_E(e)) \right)(a_{E:E_1}) \cdot \left(\bigodot_{e \in E_2} \tau_e(b_E(e)) \right)(a_{E:E_2}) \\
&\stackrel{(a)}{=} \sum_{a_{E:E_1} \in \mathcal{A}^{E_1}} f_1(a_{E:E_1}) \left(\bigodot_{e \in E_1} \tau_e(b_E(e)) \right)(a_{E:E_1}) \sum_{a_{E:E_2} \in \mathcal{A}^{E_2}} f_2(a_{E:E_2}) \left(\bigodot_{e \in E_2} \tau_e(b_E(e)) \right)(a_{E:E_2}) \\
&\stackrel{(b)}{=} \sum_{a_{E:E_1} \in \mathcal{A}^{E_1}} f_1(a_{E:E_1}) \left(\bigodot_{e \in E_1} \tau_e(b_{E:E_1}(e)) \right)(a_{E:E_1}) \sum_{a_{E:E_2} \in \mathcal{A}^{E_2}} f_2(a_{E:E_2}) \left(\bigodot_{e \in E_2} \tau_e(b_{E:E_2}(e)) \right)(a_{E:E_2}) \\
&= \langle f_1, \mathcal{E}_1^{[b_{E:E_1}]} \rangle \cdot \langle f_2, \mathcal{E}_2^{[b_{E:E_2}]} \rangle
\end{aligned}$$

where (a) is due to lemma 1 and (b) follows from the fact that $b_E(e) = b_{E:E_i}(e)$ for all $e \in E_i$, $i = 1, 2$.

Now induction can be used to show that this holds for any disjoint sets E_1, \dots, E_p . Hence, we have the following proposition (the second part of the proposition can be shown using a similar argument).

Proposition 2: Let $\{E_1, \dots, E_p\}$ be an arbitrary partition of a finite set E and \mathcal{B} be an arbitrary basis of $\mathbb{C}^{\mathcal{A}}$. Further, let $f_i \in \mathbb{C}_{\mathcal{A}}^{E_i}$ for all i . Then for any $b_E \in \mathcal{B}^E$

$$\left\langle \bigodot_{1 \leq i \leq p} f_i, \mathcal{E}^{[b_E]} \right\rangle = \prod_{1 \leq i \leq p} \langle f_i, \mathcal{E}_i^{[b_{E:E_i}]} \rangle$$

and for any $a_E \in \mathcal{A}^E$

$$\left\langle \bigodot_{1 \leq i \leq p} \hat{f}_i, \mathcal{E}_{[a_E]} \right\rangle = \prod_{1 \leq i \leq p} \langle \hat{f}_i, \mathcal{E}_{i[a_{E:E_i}]} \rangle$$

III. HOLANT THEOREM

Let E_1, \dots, E_p and E'_1, \dots, E'_r be two arbitrary partitions of a finite set E . Further let $g_i \in \mathbb{C}_{\mathcal{A}}^{E_i}$ for all $1 \leq i \leq p$ and let $h_i \in \mathbb{C}_{\mathcal{A}}^{E'_i}$ for all $1 \leq i \leq r$. The following lemma will give a proof of the Holant

theorem.

Lemma 6: Let g_1, \dots, g_p and h_1, \dots, h_r be as above. Then under any basis \mathcal{B} of $\mathbb{C}^{\mathcal{A}}$, the following holds

$$\left\langle \bigodot_{1 \leq i \leq p} g_i, \bigodot_{1 \leq i \leq r} h_i \right\rangle = \left\langle \bigodot_{1 \leq i \leq p} \hat{g}_i, \bigodot_{1 \leq i \leq r} \check{h}_i \right\rangle$$

(note that the bilinear map on the left of the equality is over the domain $\mathbb{C}_{\mathcal{A}}^E \times \mathbb{C}_{\mathcal{A}}^E$ and the one on the right is over $\mathbb{C}_{\mathcal{B}}^E \times \mathbb{C}_{\mathcal{B}}^E$).

Proof: The lemma follows via the following series of equalities:

$$\begin{aligned} \left\langle \bigodot_{1 \leq i \leq p} g_i, \bigodot_{1 \leq i \leq r} h_i \right\rangle &\stackrel{(1)}{=} \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \left(\bigodot_{1 \leq i \leq p} g_i \right)(x_E) \\ &\stackrel{(a)}{=} \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \prod_{1 \leq i \leq p} g_i(x_{E:E_i}) \\ &\stackrel{(b)}{=} \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \prod_{1 \leq i \leq p} \langle \hat{g}_i, \mathcal{E}_{[x_{E:E_i}]} \rangle \\ &\stackrel{(c)}{=} \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \left\langle \bigodot_{1 \leq i \leq p} \hat{g}_i, \mathcal{E}_{[x_E]} \right\rangle \\ &\stackrel{(1)}{=} \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \mathcal{E}_{[x_E]}(y_E) \\ &\stackrel{(d)}{=} \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \mathcal{E}(x_E, y_E) \\ &\stackrel{(d)}{=} \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \mathcal{E}^{[y_E]}(x_E) \\ &\stackrel{(1)}{=} \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \left\langle \bigodot_{1 \leq i \leq r} h_i, \mathcal{E}^{[y_E]} \right\rangle \\ &\stackrel{(c)}{=} \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \prod_{1 \leq i \leq r} \langle h_i, \mathcal{E}_i^{[y_{E:E_i}]} \rangle \\ &\stackrel{(b)}{=} \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \prod_{1 \leq i \leq r} \check{h}_i(y_{E:E_i}) \\ &\stackrel{(a)}{=} \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \left(\bigodot_{1 \leq i \leq r} \check{h}_i \right)(y_E) \\ &\stackrel{(1)}{=} \left\langle \bigodot_{1 \leq i \leq p} \hat{g}_i, \bigodot_{1 \leq i \leq r} \check{h}_i \right\rangle \end{aligned}$$

where (a), (b) and (c) are due to definition of c-tensor, definition 1 and proposition 2, respectively.

Definitions of \mathcal{E} , $\mathcal{E}_{[x_E]}$ and $\mathcal{E}^{[y_E]}$ give equalities labeled by (d). ■

Now we state and prove Holant theorem

Theorem 3: Let g_1, \dots, g_p and h_1, \dots, h_r be as defined earlier. Then for any basis \mathcal{B} of \mathbb{C}^A , the following holds

$$\sum_{x_E \in \mathcal{A}^E} \prod_{i=1}^p g_i(x_{E:E_i}) \prod_{i=1}^r h_i(x_{E:E'_i}) = \sum_{y_E \in \mathcal{B}^E} \prod_{i=1}^p \hat{g}_i(y_{E:E_i}) \prod_{i=1}^r \check{h}_i(y_{E:E'_i})$$

Proof:

$$\begin{aligned} \sum_{x_E \in \mathcal{A}^E} \prod_{i=1}^p g_i(x_{E:E_i}) \prod_{i=1}^r h_i(x_{E:E'_i}) &\stackrel{(a)}{=} \sum_{x_E \in \mathcal{A}^E} \left(\bigodot_{1 \leq i \leq p} g_i \right)(x_E) \left(\bigodot_{1 \leq i \leq r} h_i \right)(x_E) \\ &\stackrel{(1)}{=} \left\langle \bigodot_{1 \leq i \leq p} g_i, \bigodot_{1 \leq i \leq r} h_i \right\rangle \\ &\stackrel{(b)}{=} \left\langle \bigodot_{1 \leq i \leq p} \hat{g}_i, \bigodot_{1 \leq i \leq r} \check{h}_i \right\rangle \\ &\stackrel{(1)}{=} \sum_{y_E \in \mathcal{B}^E} \left(\bigodot_{1 \leq i \leq p} \hat{g}_i \right)(y_E) \left(\bigodot_{1 \leq i \leq r} \check{h}_i \right)(y_E) \\ &\stackrel{(a)}{=} \sum_{y_E \in \mathcal{B}^E} \prod_{i=1}^p \hat{g}_i(y_{E:E_i}) \prod_{i=1}^r \check{h}_i(y_{E:E'_i}) \end{aligned}$$

where (a) is due to definition of c-tensor and (b) is due to Lemma 6. ■

IV. CONCLUSION

In this paper, we provide an alternative proof of Holant theorem using a commutative notion of tensor product. The proof appears to be simpler and more transparent. As holographic algorithms are expected to demonstrate their power in many problems of information theory, we hope that this work makes this new family of algorithms and the notion of holographic reduction more accessible to audience in information theory community.

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